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Multifractal Analysis of Brownian Zero Set

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The multifractal structure of zeros Z of Brownian motion is considered. For different measures on Z we find typical characteristics: the τ-function and the multifractal spectrum $f(\alpha)$. A dimensional interpretation of $f(\alpha)$ is also discussed.

KEY WORDS: Multifractals; Brownian motion; limit theorems.

1. INTRODUCTION

The present work arose from a query of Ya. G. Sinai as to the multifractality of the set of zeros Z of Brownian motion $W(t)$, $t \geqslant 0$.

The set Z is a classical example of a stochastic fractal: it is self-similar (λZ and Z are statistically equivalent for any $\lambda > 0$) and has the Hausdorff dimension $\dim Z = 1/2$.[10] It is more difficult to define multifractality for Z, because that concept is used to characterize singular probability measures.[5,11] Roughly speaking, the measure $\mu(dt)$ on $J = [0, 1]$ is multifractal and has a multifractal spectrum $f(\alpha)$, $\alpha > 0$, if, for some sequence $\Gamma_n = \{\Delta_i\}^{(n)}$ of partitions of J, the number of partition elements of type α

$$U_-(1/|\Delta_i|) < \mu(\Delta_i)\, |\Delta_i|^{-\alpha} < U_+(1/|\Delta_i|)$$

or briefly $\mu(\Delta) \sim |\Delta|^{\alpha}$, grows like $\Delta^{-f(\alpha)} U(1/\Delta)$, where U and $U_\pm$ are slowly varying functions at ∞. Here, $\{\Gamma_n\}$ is a covering cascade, that is, Γ_{n+1} is a partition of $\Delta_i \in \Gamma_n$ and $\Delta = \max_i |\Delta_i^{(n)}| \to 0$.

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Multifractals frequently possess the following properties:

(1) The Legendre transform of $f(\alpha)$, $\mathscr{L}f = \min_\alpha (\alpha q - f(\alpha))$ is identical to the multifractal generating function $\tau(q)$. In the simplest case where the partition of J is equispaced

$$\tau(q) := \lim_{n \to \infty} \ln \left(\sum_i p_i^q \right) \Big/ \ln \Delta \tag{1}$$

where $p_i = \mu(\Delta_i^{(n)})$. For the general case see below.

(2) When f is a convex function, the value of $f(\alpha)$ coincides with the fractal dimension of the set $\{t : \mu(\Delta_t) \sim |\Delta_t|^\alpha\}$, which consists of singular points of a measure μ of type α. Here, Δ_t is a sequence of intervals containing t and $|\Delta_t| \to 0$. For example, these may be partition elements of Γ_n or any symmetrical neighborhoods of t in the ideal case.

The *partition* Γ_n, *singular points*, and the *type of fractal dimension* are all elements of the still unsettled concept of multifractality. In applications properties 1 and 2 are frequently assumed for observed multifractals and constitute the content of the so-called *multifractal formalism*. Rigorous results that corroborate the formalism for multinomial cascade measures, both deterministic and probabilistic ones, can be found in some recent publications.[2,3,9]

Two procedures are proposed to analyze the multifractality of Z. One analyzes the multifractality of a suitable measure on Z, namely the local time measure of Brownian motion:

$$L(dt) = \delta(w(t))\, dt$$

(for a rigorous definition see ref. 10). This choice can to some extent be justified by the following result due to P. Levy. Eliminate from the time axis all lacunas between zeros in Z longer than ε and define a Lebesgue measure $\mu_\varepsilon(dt)$ on the remaining intervals $\{\delta_i(\varepsilon)\} = Z_\varepsilon$, to be called ε-clusters. Then $L(dt)$ is the limit of normalized measures $c\varepsilon^{-1/2}\mu_\varepsilon(dt)$ as $\varepsilon \to 0$.

The sequence Z_ε of ε-clusters is of interest in its own in the study of Z. It can be treated as a covering cascade for Z in which the parameter ε is a resolution level for the elements of Z. In fact, any pair of points in Z less than ε apart belongs to a single ε-cluster. As is proper for coverings, the sizes of ε-clusters in a fixed finite interval J are uniformly bounded in the probabilistic sense: $\max\{|\delta_i(\varepsilon)| : \delta_i \subset J\} < \varepsilon \ln(1/\varepsilon)$ with probability $p_\varepsilon \to 1$.

The measure $L(dt)$ as the limit of $\varepsilon^{-1/2}\mu_\varepsilon(dt)$ generally distorts the information on the fine structure of ε-clusters. For this reason the second way to analyze the multifractality of Z concentrates on scaling exponents of the growth of the number of ε-clusters of size ε^{α}, $\alpha > 0$.

The multifractality of Z in both these cases expresses itself as follows. The measure $L(dt)$ has a continuous multifractal spectrum with respect to the covering cascade Z_{ε}: $f_L^*(\alpha) = 3/2 - 2\alpha$, $\alpha \in [1/2, 3/4]$, and the τ-function $\tau_L^*(q)$ such that $\tau_L^* = \mathscr{L} f_L^*$. Similarly, the number of ε-clusters of size ε^{α} in $J = [0, 1]$ grows like $\varepsilon^{-f_{CL}(\alpha)}$, where $f_{CL}(\alpha) = 1 - \alpha/2$, $\alpha \in [1, 2]$. The Legendre transform f_{CL} is identical to the τ-function of the form (1) where $p_i = |\delta_i(\varepsilon)|$ and $\Delta = \varepsilon$.

The multifractal problem of Z is treated here primarily as the modeling of a practical situation in which one studies the multifractality of a geometrical object with no prior reasons for any particular measure. Therefore all statements like limit theorems are formulated in the weak form, although the convergence for Z could be made stronger.

This paper is organized as follows: Section 2 contains some auxiliary statements that are mostly proved in the appendix; section 3 contains definitions and calculations of the τ-function for $L(dt)$ and ε-cluster size; Section 4 contains limit theorems for ε-clusters, calculations of spectral $f(\alpha)$, and some comments on the multifractal formalism. In Section 5 some extensions of the results are discussed.

Notations:

$=^d$ denotes equality in distribution.

d-lim denotes convergence in distribution; for functional objects this means the convergence of finite-dimensional distributions.

$g_{\alpha}(t)$ is a stable Levy process with independent increments and Laplace transform $E \exp[-\theta g_{\alpha}(t)] = \exp(-t\theta^{\alpha})$, $0 < \alpha < 1$.

G_{α} is the region of attraction for the stable distribution of $g_{\alpha}(1)$.

2. AUXILIARY STATEMENTS

Let $w(t)$, $t \geqslant 0$, be Brownian motion, Z the set of its zeros. The probability structure of Z can be described by local time $L(t)$.[10] The random process $L(t)$ has nondecreasing continuous paths whose growth points are statistically equivalent to zero set Z. The inverse function of $L(t)$ taken to be continuous on the right is a one-sided stable Levy process $t(L)$ with exponent 1/2. To be specific, $t(L)$ can be represented as follows:

$$t(L) = \int_0^L \int_0^{\infty} \tau \pi(dl, d\tau) \tag{2}$$

in terms of a Poissonian measure π with intensity

$$E\pi(dl, d\tau)/dl\, d\tau = (2\pi\tau^3)^{-1/2} = p(\tau)$$

Here, the variable τ describes the value of the jump $t(L)$ or the interval between zeros $w(t)$, while the variable l gives the level of local time.

Let us fix a resolution level ε on Z by connecting all pairs of points in Z less than ε apart. The resulting graph Z_ε separates into connected clusters $\delta_i(\varepsilon)$. In the one-dimensional situation considered, the clusters form intervals of lengths $|\delta_i(\varepsilon)|$. Their complement to R^1_+ consists of lacunes $\Delta_i(\varepsilon)$ of size $|\Delta_i(\varepsilon)| \geqslant \varepsilon$. The local time function is constant on Δ_i and has increments

$$L_i(\varepsilon) = \int_{\delta_i(\varepsilon)} L(dt)$$

in the cluster intervals.

Statement 1. The probability structure of the sequence of random quantities

$$S_\varepsilon = \{ |\Delta_i(\varepsilon)|, L_i(\varepsilon), |\delta_i(\varepsilon)|, i = 1,..., \nu(t, \varepsilon), t \in \delta_\nu(\varepsilon) \cup \Delta_\nu(\varepsilon) \}$$

associated with ε-clusters in $[0, t]$ is determined by the relation

$$S_\varepsilon \overset{d}{=} \{ \varepsilon\xi_i^+, (\pi\varepsilon/2)^{1/2} \eta_i, \varepsilon\xi_i^-, i = 1,..., \nu(t/\varepsilon) \}$$

where $\nu(t/\varepsilon) \geqslant 1$ is the greatest number for which

$$\sum_{i=1}^{\nu-1} (\xi_i^+ + \xi_i^-) < t/\varepsilon \tag{3}$$

Here, $(\xi_i^+, \eta_i, \xi_i^-)$ is a sequence of i.i.d. random vectors with ξ_i^+ and (η_i, ξ_i^-) independent; ξ_i^+ has the density

$$f_+(\tau) = \tfrac{1}{2}\tau^{-3/2}, \qquad \tau \geqslant 1 \tag{4}$$

while the distribution of the pair (η_i, ξ_i^-) is given by the Laplace transform

$$Ee^{-s_1\eta - s_2\xi^-} = \left[1 + s_1 - \int_0^1 (1 - e^{-s_2\tau})\, d\tau^{-1/2} \right]^{-1} \tag{5}$$

In particular, η has the exponential density

$$f_\eta(x) = \exp(-x), \qquad x > 0 \tag{6}$$

while ξ^- is such that

$$P(\xi^- > x) < c\exp(-\kappa x), \qquad \kappa = 0.567 \tag{7}$$

$$P(\xi^- < x) = \sqrt{x}/\pi[1 + o(1)], \qquad x \to 0 \tag{8}$$

i.e., ξ^- has moments $E(\xi^-)^q$ for all $q > -1/2$.

Statement 2. The distribution function $F_\rho(x)$ of the random variable

$$\eta_\rho = \eta^\rho \xi^-$$

has the following asymptotics:

(a) When $x \to 0$

$$F_\rho(x) = \begin{cases} O(x^N), & \rho < -2 \qquad \forall N \\ c_\rho x^{1/(2+\rho)}[1 + o(1)], & \rho > -2 \end{cases}$$

(b) When $x \to \infty$

$$1 - F_\rho(x) = c_\rho \begin{cases} O(x^{-N}), & \rho > 0, \quad \forall N \\ x^{2/\rho}[1 + o(1)], & -4 < \rho < 0, \quad 2/\rho \neq -1, -2,\ldots \\ x^{1/(2+\rho)}[1 + o(1)], & \rho < -4 \end{cases}$$

The analysis of multifractality involves a study of statistics of the form

$$\sum_{i=1}^{\nu(t/\varepsilon)} f(\xi_i^-, \eta_i)$$

where the number of terms depends on the terms being summed. This can be avoided by replacing $\nu(t/\varepsilon)$ by the moment $\nu^*(t/\varepsilon)$ for the first exceeding of t/ε by the sums $\sum_{i=1}^n \xi_i^+$ that are independent of $\{\xi_i^-, \eta_i\}$. The closeness of ν and ν^* is given by the following result.

Statement 3. $\nu^*(u) - \nu(u) \geqslant 0$ and

$$d\text{-}\lim_{\varepsilon \to 0} \varepsilon^\theta(\nu^*(t/\varepsilon) - \nu(t/\varepsilon)) = \begin{cases} 0, & \theta > 1/4 \\ \infty, & 0 < \theta < 1/4 \end{cases}$$

Hence, similar to $\nu(t/\varepsilon)$,

$$d\text{-lim}\ (\pi\varepsilon/2)^{1/2}\ \nu^*(t/\varepsilon) = L(t), \qquad \varepsilon \to \infty$$

Statement 4. Let $\zeta_i = f(\xi_i^-, \eta_i) \geqslant 0$. If

$$P(\zeta_i > x) = \frac{c}{\Gamma(1-\alpha)} x^{-\alpha}[1+o(1)], \qquad x \to \infty, \quad \alpha \in (0, 1) \tag{9}$$

then

$$d\text{-}\lim_{\varepsilon \to 0} \varepsilon^{1/2\alpha} \sum_{i=1}^{\nu(t/\varepsilon)} \zeta_i [\sqrt{\varepsilon}\, \nu(t/\varepsilon)]^\beta = g_\alpha((2/\pi)^{1/2} cL(t))[(2/\pi)^{1/2} L(t)]^\beta \tag{10}$$

where $g_\alpha(u)$ is a stable Levy process of index α which is independent of $L(t)$.

If $m = E\zeta_i < \infty$, then

$$d\text{-}\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \sum_{i=1}^{\nu(t/\varepsilon)} \zeta_i [\sqrt{\varepsilon}\, \nu(t/\varepsilon)]^\beta = m[(2/\pi)^{1/2} L(t)]^{1+\beta} \tag{11}$$

The proofs of Statements 1–4 are relegated to the appendix.

Statement 5. Let $\varphi(\varepsilon) \uparrow \infty$, $\varepsilon \to 0$. Then for a fixed t

$$P\{\varepsilon^2/\varphi(\varepsilon) < |\delta_i(\varepsilon)| < \varepsilon \ln(1/\varepsilon), i = 1,..., \nu(t, \varepsilon)\} \to 1, \qquad \varepsilon \downarrow 0 \tag{12}$$

Proof. Denote by $\bar{\delta}_\varepsilon$ and $\underline{\delta}_\varepsilon$ the greatest and smallest of

$$\{|\delta_i(\varepsilon)|, i = 1,..., \nu(t, \varepsilon)\}$$

respectively. Let $A_\varepsilon = \{\nu_\varepsilon < \varepsilon^{-\kappa'}\}$, $\nu_\varepsilon = \nu(t/\varepsilon)$. Then

$$\begin{aligned} P\{\bar{\delta}_\varepsilon > x\} &\leqslant P\{\max(\xi_i^-, i = 1,..., \nu_\varepsilon) > x/\varepsilon, A_\varepsilon\} + P(\bar{A}_\varepsilon) \\ &\leqslant \varepsilon^{-\kappa'} P(\xi_i^- > x/\varepsilon) + P(\bar{A}_\varepsilon) \end{aligned}$$

Take $\kappa' \in (1/2, \kappa)$, $\kappa = 0.576$. Then (7) gives

$$P\{\bar{\delta}_\varepsilon > \varepsilon \ln(1/\varepsilon)\} < c\varepsilon^{\kappa - \kappa'} + P(\bar{A}_\varepsilon) = o(1)$$

In fact, $P(\bar{A}_\varepsilon) \to 0$, $\varepsilon \to 0$, since $\nu_\varepsilon \sqrt{\varepsilon}/\varphi(\varepsilon) \to^d 0$ if $\varphi(\varepsilon) \to \infty$ with $\varepsilon \to 0$. Similarly, using (8), one has $P(\underline{\delta}_\varepsilon < \varepsilon^2/\varphi(\varepsilon)) = o(1)$, $\varepsilon \to 0$. Combining both estimates, we have (12).

3. MULTIFRACTAL GENERATING FUNCTION $\tau(q)$

One finds in refs. 6 and 7 extension of the definition of the τ-function to a partition of the measure support with unequal cells. Following along these lines, we define $\tau(q)$ for $L(dt)$ on $J = [0, 1]$. Consider ε-clusters as partition elements Γ_ε of the interval J. The partition outside Z_ε need not be defined, since the L measure of the complement of ε-clusters is zero. Consider the function

$$\Phi_\varepsilon(q, \tau) = \sum_i l_i^q(\varepsilon)\, |\delta_i(\varepsilon)|^{-\tau}, \qquad \delta_i(\varepsilon) \subset J \tag{13}$$

where

$$l_i(\varepsilon) = L_i(\varepsilon) \Big/ \sum_j L_j(\varepsilon) \qquad \delta_i(\varepsilon) \subset J$$

are normalized increments of local time on ε-clusters. If τ^* is such that the following limits exist:

$$d\text{-}\lim_{\varepsilon \to 0} \Phi_\varepsilon(q, \tau) = \begin{cases} \infty, & \tau > \tau^* \\ 0, & \tau < \tau^* \end{cases} \tag{14}$$

then $\tau_L^*(q)\colon q \to \tau^*$ is a multifractal generating function of the measure $L(dt)$ or the τ-function of singularities of $L(dt)$.

The definition of τ^* requires the cells with nonzero increments of $L(dt)$ to be uniformly small. Statement 5 shows that the sizes of ε-clusters in a fixed finite interval are uniformly small in a stochastic sense:

$$P\{|\delta_i(\varepsilon)| \in (\varepsilon^2/\ln(1/\varepsilon), \varepsilon \ln(1/\varepsilon)), \forall \delta_i \in J\} \to 1, \qquad \varepsilon \to 0$$

If a τ-function of (1) exists, it is identical with the generalized function $\tau^*(q)$ for covers Γ_n with equal cells. The original definition (1) is meaningful for the general partitions, too. Below, (1) is used to define the τ-function for the size of increments of $L(dt)$ on ε-clusters, as well as for the size of the ε-clusters themselves.

Theorem 1. Consider ε-clusters on the interval (0, 1). Then:

(a) The τ-function of singularities of $L(dt)$ is

$$\tau_L^*(q) = \tfrac{1}{2} \min(q-1, 3/2q), \qquad |q| < \infty$$

(b) The τ-function of the size of increments of $L(dt)$ on ε-clusters is

$$\tau_L(q) := d\text{-}\lim_{\varepsilon \to 0} \ln \sum_{i=1}^{\nu_\varepsilon} L_i^q(\varepsilon)/\ln \varepsilon = \tfrac{1}{2} \min(q-1, 2q), \qquad |q| < \infty$$

(c) The τ-function of ε-clusters size is

$$\tau_{CL}(q) := d\text{-}\lim_{\varepsilon \to 0} \ln \sum_{i=1}^{\nu_\varepsilon} |\delta_i(\varepsilon)|^q/\ln \varepsilon = \min(q-1/2, 2q), \qquad |q| < \infty$$

Here, ν_ε is the number of ε-clusters in $[0, 1]$.

Proof. (a) A term of (13) is

$$l_i^q(\varepsilon)\,|\delta_i(\varepsilon)|^{-\tau} \overset{d}{=} \varepsilon^{-\tau}\eta_i^q[\xi_i^-]^{-\tau}\left[\sum_{i=1}^{\nu_\varepsilon}\eta_i\right]^{-q}$$

Since

$$\zeta=\eta^q[\xi_i^-]^{-\tau}=[\eta^\rho\xi_i^-]^{-\tau}, \qquad \rho=-q/\tau$$

one can use Statement 2 to find the asymptotics of $P(\zeta>x)$ as $x\to\infty$. It is easy to see that

$$P(\zeta>x)=cx^{-\alpha}[1+o(1)], \qquad x\to\infty, \quad 0<\alpha<1$$

where

$$\alpha=\begin{cases}(2\tau-q)^{-1}, & \tau>\max(q/4,(q+1)/2)\\ -2/q, & \tau<q/4, \quad q<-2, \quad 2\tau\neq qn, \quad n=1,2,\ldots\end{cases} \tag{15}$$

In the parameter region (q,τ)

$$D=\{(q,\tau):\tau<(q+1)/2,\, q>-2\} \tag{16}$$

the quantity ζ has a finite mean $E\zeta<\infty$.

It remains to use Statement 4 for deriving the limiting distribution (13).

One has

$$\Phi_\varepsilon(q,\tau) \overset{d}{=} \varepsilon^{-\tau}\sum_{i=1}^{\nu_\varepsilon}\eta_i^q[\xi_i^-]^{-\tau}\,\nu_\varepsilon^{-q}\left[\nu_\varepsilon^{-1}\sum_{i=1}^{\nu_\varepsilon}\eta_i\right]^{-q}$$

If $\alpha=\alpha(\tau,q)$ is given by (15), then

$$\varepsilon^{1/(2\alpha)}\sum_{i=1}^{\nu_\varepsilon}\eta_i^q[\xi_i^-]^{-\tau}[\sqrt{\varepsilon}\,\nu_\varepsilon]^{-q} \xrightarrow{d} g_\alpha(\hat{c}L(1))[(2/\pi)^{1/2}L(1)]^{-q}=X$$

where $g_\alpha(\cdot)$ and $L(1)$ are independent, and $\hat{c}=c\Gamma(1-\alpha)(2/\pi)^{1/2}$. For all parameters (q,τ)

$$\nu_\varepsilon^{-1}\sum_{i=1}^{\nu_\varepsilon}\eta_i \xrightarrow{d} E\eta=1$$

hence

$$\varepsilon^{\tau+(\alpha^{-1}-q)/2}\Phi_\varepsilon(q,\tau) \xrightarrow{d} X \tag{17}$$

If $(q, \tau) \in D$, then according to Statement 4,

$$\varepsilon^{\tau+(\alpha^{-1}-q)/2}\Phi_\varepsilon(q,\tau) \xrightarrow{d} E\zeta\cdot[(2/\pi)^{1/2}L(1)]^{1-q}, \qquad \varepsilon\to 0 \tag{18}$$

Relations (17) and (18) determine the asymptotics of $\Phi_\varepsilon(q, \tau)$ for all parameters (q, τ) except for a countable sequence of rays:

$$2\tau - q = 1, \quad q \geqslant -2; \qquad 4\tau - nq = 0, \quad q < -2, \quad n = 1, 2, 4, 6,...$$

The functions $\tau \to \Phi_\varepsilon(q, \tau)$ are monotonic, hence the determination of $\tau^*(q)$ just requires knowledge of the asymptotics of $\Phi_\varepsilon(q, \tau)$ for a parameter set (q, τ) which is everywhere dense. The limit relations (17) and (18) yield, in accordance with (14), equations for τ^*:

$$\begin{cases} \tau + (1/\alpha - q)/2 = 0, & (q,\tau)\,\bar{\in}\, D \\ \tau + (1-q)/2 = 0, & (q,\tau)\in D \end{cases} \tag{19}$$

where α and D are given by (15) and (16), respectively. The value $\alpha^{-1} = 2\tau - q$ produces a contradiction for all q, while $\alpha^{-1} = -q/2$ gives the desired form $\tau_L^*(q)$ for all $q \leqslant -2$. When $q \geqslant -2$, the function $\tau_L^*(q)$ is given by the second equation of (19). The other statements can be proved in a similar fashion.

4. THE MULTIFRACTALITY OF Z

4.1. Scaling Exponents

Theorem 2. Let $N_\varepsilon^{(\alpha)}(t)$ be the number of ε-clusters in $(0, t)$ of type α, that is, clusters that obey one of the fixed requirements

$$\begin{aligned} &(\mathrm{a}) \quad |\delta_i(\varepsilon)|^\alpha \varphi(|\delta_i(\varepsilon)|) < L_i(\varepsilon) < x\,|\delta_i(\varepsilon)|^\alpha \\ &(\mathrm{b}) \quad \varepsilon^\alpha\varphi(\varepsilon) < L_i(\varepsilon) < x\varepsilon^\alpha \\ &(\mathrm{c}) \quad \varepsilon^\alpha\varphi(\varepsilon) < |\delta_i(\varepsilon)| < x\varepsilon^\alpha \end{aligned} \tag{20}$$

where $\varphi \geqslant 0$ is a nondecreasing function which is continuous at 0, $\varphi(0) = 0$, and

$$\varphi(x)\,x^{-p} \to \infty, \qquad x \to 0 \quad \forall p > 0$$

In that case one has convergence of the following random processes:

$$d\text{-}\lim_{\varepsilon\to 0} N_\varepsilon^{(\alpha)}(t)\,\varepsilon^{f(\alpha)} = \begin{cases} C_{\alpha,x}L(t), & \alpha\in[\alpha_1,\alpha_2) \\ \Pi_\Lambda(t), & \Lambda = \lambda_x L(t), \quad \alpha = \alpha_2 \end{cases}$$

where $\Pi_A(t)$ is a Poissonian process with independent random intensity measure $dA(t)$; $f(\alpha)$, the interval $[\alpha_1, \alpha_2]$, and λ_x are given by

$$\begin{array}{llll} \text{(a)} & f_L^*(\alpha) = 3/2 - 2\alpha, & \alpha \in [1/2, 3/4], & \lambda_x = cx^2 \\ \text{(b)} & f_L(\alpha) = 1 - \alpha, & \alpha \in [1/2, 1], & \lambda_x = 2/\pi \cdot x \\ \text{(c)} & f_{CL}(\alpha) = 1 - \alpha/2, & \alpha \in [1, 2], & \lambda_x = (2x/\pi^3)^{1/2} \end{array}$$

When $\alpha \bar{\in} [\alpha_1, \alpha_2]$, one has d-lim $N_\varepsilon^{(\alpha)}(t) = 0$.

The Legendre transforms $\mathscr{L}f$ of these functions $f(\alpha)$ are identical with the respective τ-functions from Theorem 1.

Remarks. (i) Theorem 2 shows that the number of ε-clusters of type α on a fixed finite interval has a power law of growth in ε: $O(\varepsilon^{-f(\alpha)})$. Outside of this interval of α the exponent $f(\alpha)$ should naturally be defined as $f(\alpha) = -\infty$, since $P(N_\varepsilon^{(\alpha)}(t_0) = 0) \to 1$, $\varepsilon \to 0$. The function $f(\alpha)$ thus defined gives the multifractal spectrum of (a) singularities of $L(dt)$, (b) increments of $L(dt)$ on ε-clusters, and (c) the size of ε-clusters. The fact that $\mathscr{L}f$ is identical with the respective τ-functions means that the first requirement of a multifractal formalism for the multifractal characteristics of Z is fulfilled.

(ii) The relation $dN_\varepsilon^{(\alpha)}(t) \simeq c\varepsilon^{-f(\alpha)}\, dL(t)$, $1 < \alpha < 2$, $\varepsilon \to 0$, for ε-clusters is an extension (in the weak sense) of Levy's result[10] for gaps of size $\geqslant \varepsilon$ on Z. (The number of such gaps on $[0, t]$ differs from that of ε-clusters by no more than 1.) This relation also shows that the local time measure is quite a suitable tool to study the fine structure of Z. This could not be deduced from Levy's result alone. It is for this reason that our analysis is concerned with two objects, the dL measure and ε-clusters.

Proof. Let $A_i^{(\alpha)}$ be events of type (20).

Step 1. Evaluation of the probability $P(A_i^{(\alpha)})$. The event (20a) is equivalent to the event

$$A_i^{(\alpha)}\!: \quad c(\varepsilon\xi_i^-)^\alpha\, \varphi(\varepsilon\xi_i^-) < \varepsilon^{1/2}\eta_i < (\varepsilon\xi_i^-)^\alpha\, cx, \qquad c = (2/\pi)^{1/2}$$

Now evaluate the probability of $A_i^{(\alpha)}$:

$$\begin{aligned} P(A_i^{(\alpha)}) &= P\{\eta_i[\xi_i^-]^{-\alpha} < cx\varepsilon^{\alpha - 1/2}\} - P\{\eta_i[\xi_i^-]^{-\alpha} < c\varphi(\varepsilon\xi_i^-)\,\varepsilon^{\alpha - 1/2}\} \\ &= P_1(\varepsilon) - P_2(\varepsilon) \end{aligned}$$

The use of Statement 2 yields the asymptotics of P_1 as $\varepsilon \to 0$:

$$P_1(\varepsilon) = \begin{cases} 1 - o(\varepsilon^{-N}) \quad \forall N, & \alpha \in (0, 1/2) \\ P(\eta/(\xi^-)^{1/2} < cx), & \alpha = 1/2 \\ C_\alpha x^2 \varepsilon^{2\alpha - 1}[1 + o(1)], & \alpha > 1/2, \quad \alpha \neq 2, 3,... \end{cases} \tag{21}$$

Since φ is monotonic,

$$\varphi(\varepsilon\xi^-) < \varphi(\varepsilon^{1-\rho}) \qquad \text{if} \quad \xi^- < \varepsilon^{-\rho}, \qquad \rho \in (0, 1)$$

one has

$$P_2(\varepsilon) \leqslant P\{\eta(\xi^-)^{-\alpha} < c\varepsilon^{\alpha - 1/2}\varphi(\varepsilon^{1-\rho})\} + P\{\xi^- > \varepsilon^{-\rho}\} \tag{22}$$

Let $\alpha \geqslant 1/2$. Then the first term in (22) is $o(\varepsilon^{2\alpha - 1})$. This follows from (21) with $x = \varphi(\varepsilon^{1-\rho}) \to 0$, $\varepsilon \to 0$. The second term in (22) can be evaluated by using (7):

$$P(\xi^- > \varepsilon^{-\rho}) \leqslant c \exp(-\kappa\varepsilon^{-\rho})$$

Hence $P_2 = o(\varepsilon^{2\alpha - 1})$, $\alpha \geqslant 1/2$.

Let $\alpha < 1/2$. One has

$$\begin{aligned} P(A_i^{(\alpha)}) &\leqslant P\{\eta(\xi^-)^{-\alpha} > c\varepsilon^{\alpha - 1/2}\varphi(\varepsilon\xi^-)\} \\ &\leqslant P\{\eta(\xi^-)^{-\alpha} > c\varepsilon^{\alpha - 1/2}\varphi(\varepsilon^{1+n})\} + P\{\xi^- < \varepsilon^n\} \\ &\leqslant P\{\eta(\xi^-)^{-\alpha} > c\varepsilon^{(\alpha - 1/2)/2}\} + P\{\xi^- < \varepsilon^n\}, \qquad \forall n > 0, \quad \varepsilon < \varepsilon_0 \end{aligned} \tag{23}$$

The last inequality uses the fact that

$$\varepsilon^{-c_1}\varphi(\varepsilon^{c_2}) \to \infty, \qquad \varepsilon \to 0, \quad c_1 = (\alpha - 1/2)/2, \quad c_2 = 1 + n$$

In virtue of (21) the first term in (23) is $O(\varepsilon^{-N})$, $\forall N$. The same is true for the second term because of (8) and the arbitrariness of n. Hence, in case (a),

$$P(A_i^{(\alpha)}) = \begin{cases} o(\varepsilon^N), & \alpha \in (0, 1/2), \quad \forall N \\ P_1(\varepsilon)[1 + o(1)], & \alpha \geqslant 1/2, \quad \alpha \neq 2, 3,... \end{cases} \tag{24}$$

Step 2. $N_\varepsilon^{(\alpha)}(t) \to^d 0$, when $\alpha \bar{\in} [\alpha_1, \alpha_2]$. We continue with case (a). We have $(\alpha_1, \alpha_2) = (1/2, 3/4)$. Represent $N_\varepsilon^{(\alpha)}(t)$ in the form

$$N_\varepsilon^{(\alpha)}(t) = \sum_{i=1}^{\nu(t/\varepsilon)} \chi_i \tag{25}$$

where χ_i is the characteristic function of event $A_i^{(\alpha)}$.

According to (24) and (21),

$$E\chi_i = P(A_i^{(\alpha)}) = O(\varepsilon^{1/2+\delta}), \qquad \delta = \delta(\alpha) > 0$$

Let

$$B_n = \left\{\omega : \sum_{i=1}^{n} \chi_i > 1/2\right\}, \qquad n_\varepsilon = \varepsilon^{-1/2-\rho}, \qquad 0 < \rho < \delta$$

Then

$$\begin{aligned} P\{N_\varepsilon^{(\alpha)}(t) > 1/2\} &= P\{B_{\nu(t/\varepsilon)}, \nu(t/\varepsilon) \leqslant n_\varepsilon\} + P\{\nu(t/\varepsilon) > n_\varepsilon\} \\ &< P\{B_{n_\varepsilon}, \nu(t/\varepsilon) \leqslant n_\varepsilon\} + o(1) \leqslant P(B_{n_\varepsilon}) + o(1) \end{aligned}$$

From Chebyshev's inequality one gets

$$P(B_{n_\varepsilon}) < 2n_\varepsilon P(A_i^{(\alpha)}) = O(\varepsilon^{\delta-\rho}) = o(1), \qquad \varepsilon \to 0$$

hence

$$P(\max_{[0,T]} N_\varepsilon^{(\alpha)}(t) > 1/2) = P(N_\varepsilon^{(\alpha)}(T) > 1/2) = o(1), \qquad \varepsilon \to 0$$

Step 3. The limit of $\varepsilon^{f(\alpha)} N_\varepsilon^{(\alpha)}(t)$ *as* $\alpha \in [\alpha_1, \alpha_2]$. Because of (25), the limiting distribution of $N_\varepsilon^{(\alpha)}(t)$ can be found similar to the limiting distribution of (10) from Statement 4. The difference is that χ_i is a function of ξ_i^-, η_i, and the parameter ε, which by no means affects the method of proof. We begin by considering instead of (25) the modified sum

$$\xi_\varepsilon^*(t) = \sum_{i=1}^{\nu^*(t/\varepsilon)} \chi_i \varepsilon^{f(\alpha)}$$

where $\nu^*(t/\varepsilon)$ is independent of $\{A_i^{(\alpha)}\}$.

The Laplace transform of the distribution of $\xi_\varepsilon^*(t_1) \cdots \xi_\varepsilon^*(t_N)$ is

$$\Phi_\varepsilon = E\exp\left(-\sum_{i=1}^{N} \theta_i \xi_\varepsilon^*(t_i)\right) = E\exp\left(-\sum_{k=1}^{N} \psi_{\varepsilon,k}\, \Delta\nu_k^*\right)$$

where

$$\Delta\nu^*(k) = \nu^*(t_k/\varepsilon) - \nu^*(t_{k-1}/\varepsilon)$$

$$\psi_{\varepsilon,k} = -\ln\left\{1 - \left[1 - \exp\left(-\sum_{i=k}^{N} \theta_i \varepsilon^{f(\alpha)}\right)\right] P(A^{(\alpha)})\right\}$$

In the case (a)

$$\begin{aligned} f(\alpha) &= 3/2 - 2\alpha, \qquad (\alpha_1, \alpha_2) = (1/2, 3/4) \\ P(A^{(\alpha)}) &= C_{\alpha,x}\varepsilon^{1/2 - f(\alpha)}[1 + o(1)] \end{aligned} \tag{26}$$

where

$$C_{\alpha,x} = \begin{cases} P(\eta/(\xi^-)^{1/2} < (2/\pi)^{1/2}), & \alpha = 1/2 \\ C_\alpha x^2, & \alpha > 1/2 \end{cases} \tag{27}$$

Let $\alpha \in [\alpha_1, \alpha_2)$. Then $f(\alpha) > 0$ and

$$\psi_{\varepsilon,k} = C_{\alpha,x} \sum_{i=k}^{N} \theta_i \varepsilon^{1/2}[1 + o(1)]$$

where $o(1) \to 0$, $\varepsilon \to 0$ uniformly over θ_i in $[0, \theta]$, $\theta < \infty$. Because

$$\varepsilon^{1/2}\, \Delta v_k^* \xrightarrow{d} (2/\pi)^{1/2}\, [L(\tau_k) - L(t_{k+1})], \qquad \varepsilon \to 0$$

one gets

$$\psi_\varepsilon := \sum_{k=1}^{N} \psi_{\varepsilon,k}\, \Delta v_k^* \xrightarrow{d} (2/\pi)^{1/2}\, C_{\alpha,x} \sum_{i=1}^{N} \theta_i L(t_i) =: \psi_0, \qquad \varepsilon \to 0$$

Recalling that $\exp(-\psi_\varepsilon) < 1$, one gets

$$\Phi_\varepsilon = E \exp(-\psi_\varepsilon) \to E \exp(-\psi_0), \qquad \varepsilon \to 0$$

hence

$$d\text{-lim}\, \xi_\varepsilon^*(t) = (2/\pi)^{1/2}\, C_{\alpha,x} L(t)$$

Let $\alpha = \alpha_2$. Then $f(\alpha) = 0$ and

$$\psi_{\varepsilon,k} = \left[1 - \exp\left(- \sum_{i=k}^{N} \theta_i \right) \right] C_{\alpha_2,x}\varepsilon^{1/2}[1 + o(1)]$$

so

$$\psi_\varepsilon \xrightarrow{d} (2/\pi)^{1/2}\, C_{\alpha_2,x} \sum_{k=1}^{N} \left[1 - \exp\left(- \sum_{i=k}^{N} \theta_i \right) \right] [L(t_k) - L(t_{k-1})]$$

and

$$d\text{-}\lim_{\varepsilon\to 0} \xi_\varepsilon^*(t) = \Pi_\Lambda(t)$$

where Π_Λ is a Poissonian process with the random intensity measure

$$d\Lambda(t) = C_{\alpha_2, x}(2/\pi)^{1/2}\, dL(t)$$

that is independent of events Π.

In the case (a),

$$C_{\alpha_2, x} = C_{\alpha_2} x^2$$

Recalling that $\nu^*(t/\varepsilon)$ and $\nu(t/\varepsilon)$ have similar values (Statement 3), we show that

$$\Delta_\varepsilon(t) = \xi_\varepsilon^*(t) - N_\varepsilon^{(\alpha)}(t)\, \varepsilon^{f(\alpha)} \xrightarrow{d} 0$$

This can be seen as follows. One has

$$P(\Delta_\varepsilon(t) > y) = P\left(\sum_{i=1}^{\nu^*-\nu} \chi_i \varepsilon^{f(\alpha)} > y\right), \qquad \nu^* = \nu^*(t/\varepsilon), \quad \nu = \nu(t/\varepsilon)$$

where the moment ν is Markovian. For this reason the quantities $\chi_{\nu+i}$ in the last relation can be replaced by χ_i such that $\nu^* - \nu$ is independent of $A_i^{(\alpha)}$. But

$$\begin{aligned}
P\left\{\sum_{i=1}^{\nu^*-\nu} \chi_i \varepsilon^{f(\alpha)} > y\right\} &< P\left\{\sum_{i=1}^{\nu^*-\nu} \chi_i \varepsilon^{f(\alpha)} > y,\ \nu^* - \nu < \varepsilon^{-\theta}\right\} + P\{\nu^* - \nu > \varepsilon^{-\theta}\} \\
&< P\left\{\sum_{i=1}^{\varepsilon^{-\theta}} \chi_i \varepsilon^{f(\alpha)} > y\right\} + P\{\nu^* - \nu > \varepsilon^{-\theta}\} \\
&< \varepsilon^{f(\alpha)-\theta} P(A^{(\alpha)})/y + P(\nu^* - \nu > \varepsilon^{-\theta})
\end{aligned} \tag{28}$$

Chebyshev's inequality has been used here. Let $1/4 < \theta < 1/2$; then the first term in (28) is $O(\varepsilon^{1/2-\theta})$ [see (26)], while the second is $o(1)$ because of Statement 4. Since y is arbitrary, one has

$$d\text{-}\lim_{\varepsilon\to 0} \Delta_\varepsilon(t) = 0$$

Step 4. Cases (b), (c). The proof repeats the preceding steps. The asymptotics of $P(A^{(\alpha)})$ requires specification. The interval $[\alpha_1, \alpha_2]$ is determined by values of α such that $N_\varepsilon^{(\alpha)}(t) \to^d 0$. This requirement is equivalent to $\nu(t/\varepsilon)\, P(A^{(\alpha)}) \to 0$ or $\varepsilon^{-1/2} P(A^{(\alpha)}) \to 0$, $\varepsilon \downarrow 0$.

Case (b):

$$P(A^{(\alpha)}) = P\{(2/\pi)^{1/2}\,\varepsilon^{\alpha-1/2}\varphi(\varepsilon) \leqslant \eta_i \leqslant x(2/\pi)^{1/2}\,\varepsilon^{\alpha-1/2}\}$$

Using (6), we have

$$P(A^{(\alpha)}) = \begin{cases} O(\varepsilon^N) \quad \forall N, & 0<\alpha<1/2 \\ 1-\exp[-(2/\pi)^{1/2}\,x]+o(1), & \alpha=1/2 \\ (2/\pi)^{1/2}\,\varepsilon^{1/2-f(\alpha)}[1+o(1)], & \alpha>1/2 \end{cases}$$

where $f(\alpha) = 1-\alpha$. Hence $(\alpha_1, \alpha_2) = (1/2, 1)$ and

$$A(t) = x(2/\pi)^{1/2}\,(2/\pi)^{1/2}\,L(t) = x\cdot 2/\pi\cdot L(t)$$

Case (c):

$$P(A^{(\alpha)}) = P(\varepsilon^{\alpha-1}\varphi(\varepsilon) < \xi^- \leqslant x\varepsilon^{\alpha-1})$$

Using (7) and (8), we have

$$P(A^{(\alpha)}) = \begin{cases} O(\varepsilon^N) \quad \forall N, & 0<\alpha<1 \\ P(\xi^- \leqslant x)+o(1), & \alpha=1 \\ \sqrt{x}/\pi\varepsilon^{1/2-f(\alpha)}[1+o(1)], & \alpha>1 \end{cases}$$

where $f(\alpha) = 1-\alpha/2$. Hence $(\alpha_1, \alpha_2) = (1, 2)$ and

$$A(t) = (\sqrt{x}/\pi)(2/\pi)^{1/2}\,L(t) = (2x/\pi^3)^{1/2}\,L(t)$$

4.2. Scaling Exponents and Dimensions

Let $\delta_i^{(\alpha)}(\varepsilon)$ be type α ε-clusters defined by any of (a)–(c) in Theorem 2. The second property of the multifractal formalism for $L(dt)$ must make $f_L^*(\alpha)$ identical to the dimension $D(\alpha)$ of a suitable limit $Z_0^{(\alpha)}$ for the sets

$$Z_\varepsilon^{(\alpha)} = \bigcup_i \{\delta_i^{(\alpha)}(\varepsilon)\}$$

The dimensional interpretation of $f(\alpha)$ in cases when this is not related to measure requires some specification. The case (c) concerns abnormally small ε-clusters of size $\delta \sim \varepsilon^\alpha$. The number of these increases like $\varepsilon^{-f_{CL}(\alpha)} = \delta^{-f_{CL}(\alpha)/\alpha}$, so the function $D_{CL}(\alpha) = f_{CL}(\alpha)/\alpha$, $\alpha \in [1, 2]$, should now be the dimension.

In all cases considered a suitable dimension of $Z_0^{(\alpha)}$ may be represented by $D(\alpha)$ as found from the following relation:

$$d\text{-}\lim_{\varepsilon \to 0} \sum_{\delta_i \subset J} |\delta_i^{(\alpha)}(\varepsilon)|^p = \begin{cases} 0, & p > D(\alpha) \\ \infty, & p < D(\alpha) \end{cases}$$

This is consistent with the preceding definition of $D(\alpha)$, producing $D_L(\alpha) = f_L(\alpha)/(2\alpha)$, $\alpha \in [1/2, 1]$. The result can be easily derived using Statements 1 and 3. Note that the range of $D(\alpha)$ is the same ($[0, 1/2]$) in all cases.

Theorem 2 provides an instructive example of a limit for subsets of $Z_\varepsilon^{(\alpha)}$. The process $t \to N_\varepsilon^{(\alpha)}(t)$ generates a counting measure $dN_\varepsilon^{(\alpha)}(t)$ for ε-clusters of type α, the support consisting of the rightmost points of the clusters $\delta_i^{(\alpha)}(\varepsilon)$. The convergence of finite-dimensional distributions of the processes $\varepsilon^{f(\alpha)} N_\varepsilon^{(\alpha)}(t)$ with nondecreasing paths entails weak convergence of the measures $\varepsilon^{f(\alpha)}\, dN_\varepsilon^{(\alpha)}(t)$,[1] i.e., one has

$$\int \varphi(t)\, dN_\varepsilon^{(\alpha)}(t)\, \varepsilon^{f(\alpha)} \xrightarrow{d} \int \varphi(t)\, d\mu^{(\alpha)}(t), \qquad \varphi \in C_0(R^+) \tag{29}$$

for finite continuous functions, where the limiting measure $\mu^{(\alpha)}$ is proportional to $dL(t)$ or to a Poissonian measure with random intensity measure $cL(t)$. This means that the limiting measure is concentrated in Z and is stochastically continuous. For this reason (29) can be extended to finite bounded functions. Hence we have the following result.

Corollary to Theorem 2. Let $[\alpha_1, \alpha_2]$ be the range of the spectral parameter α indicated for various singularities in Z in Theorem 2. Then the limiting measure support $Z_0^{(\alpha)}$ for $\varepsilon^{f(\alpha)}\, dN_\varepsilon^{(\alpha)}(t)$ is statistically equivalent to Z. Hence

$$\dim Z_0^{(\alpha)} = 1/2 > D(\alpha), \qquad \alpha \in [\alpha_1, \alpha_2]$$

Different limits for $Z_\varepsilon^{(\alpha)}$ for a discrete ε were recently examined by D. Dolgopyat (personal communication). If $\varepsilon_n = c^{-n}$, $c > 1$, then the lower (Z_-) and upper (Z_+) limits are significantly different: $\dim Z_- = 0$, while $\dim Z_+$ coincides with the above values of $D(\alpha)$. When the resolution parameter ε undergoes a superrapid decrease, $\varepsilon_n/\varepsilon_{n+1} \to \infty$, then the dimensions of Z_- and Z_+ are identical and are equal to $D(\alpha)$. These results point to a purely mathematical character of the second property of the multifractal formalism for Z.

5. GENERALIZATIONS

The central role in the study of zeros of Brownian motion is played by the connection of Z with the Levy process $t(L)$ possessing homogeneous independent one-sided increments and jump density[4]

$$p(\tau) = c\tau^{-\beta-1}, \qquad \tau \geqslant 0, \quad \beta = 1/2$$

The set Z was identified with a closure of the range of values for $t(L)$. When Z is treated in this manner, all results derived here are immediately relevant to Levy processes with any index $\beta \in (0, 1)$. In particular, the spectral function of ε-clusters is $f_{CL}(\alpha) = (2-\alpha)\beta$, $\alpha \in [1, 2]$, while that of $L(dt)$ is $f_L^*(\alpha) = 3\beta - 2\alpha$, $\alpha \in [\beta, \frac{3}{2}\beta]$. Here, $L[0, t]$ is a modified continuous inverse function of $t(L)$.

Setting

$$p(\tau) = 2\pi^{-1/2} \exp(-\tau)[1 - \exp(-2\tau)]^{-3/2}, \qquad \tau > 0$$

one gets a process[8] $t(L)$ which is the inverse of the local time of the Ornstein–Uhlenbeck process $x(t)$:

$$dx(t) + x(t)\, dt = dw(t), \qquad x(0) = 0, \quad t \geqslant 0$$

Therefore, the zeros of $x(t)$ can be studied in the same manner as were those of Brownian motion.

APPENDIX

Proof of Statement 1. Let $\delta_i(\varepsilon)$ denote ε-clusters, $\Delta_i(\varepsilon)$ the intervals between ε-clusters, and $L_i(\varepsilon)$ increments of local time $L(t)$ on $\delta_i(\varepsilon)$. Let $t(L)$ be the inverse function of $L(t)$ and (2) be its representation through the Poissonian measure $\pi(dl, d\tau)$ with intensity $(2\pi\tau^{-3})^{-1/2} = p(\tau)$. From this it follows that the jumps in $t(L)$ greater than ε, these being the $\Delta_i(\varepsilon)$ intervals, are arranged as follows: the times of jumps l_i on the L axis form the Poissonian process

$$\pi(L) = \int_0^L \int_\varepsilon^\infty \pi(dl, d\tau), \qquad \pi(0) = 0$$

with intensity

$$\Lambda_\varepsilon = \int_\varepsilon^\infty p(\tau)\, d\tau = (2/\pi)^{1/2}\, \varepsilon^{-1/2}$$

The jump sizes $|\Delta_i(\varepsilon)|$ are mutually independent, and do not depend on l_i and $\delta_i(\varepsilon)$ associated with π in the interval $0 < \tau < \varepsilon$.

The density of $|\Delta_i|$ is

$$p(\tau)/\Lambda_\varepsilon = \tfrac{1}{2}\sqrt{\varepsilon}\,\tau^{-3/2}, \qquad \tau > \varepsilon$$

On normalizing $|\Delta_i| = \varepsilon\xi_i^+$, one arrives at (4).

The quantities $l_i - l_{i-1}$ give increments of local time $L_i(\varepsilon)$ in cluster intervals $\delta_i(\varepsilon)$. Since $\{l_i\}$ are points of jumps for $\pi(L)$, the $L_i(\varepsilon)$ are independent and exponentially distributed with parameter Λ_ε. On normalizing $L_i = (\pi\varepsilon/2)^{1/2}\,\eta_i$, one arrives at (6).

The cluster size is

$$|\delta_i(\varepsilon)| = \int_0^\varepsilon \tau\pi[(l_{i-1}, l_i), d\tau]$$

Because π is Poissonian, the $(L_i, |\delta_i|)$ are independent for different i. Let $l_i - l_{i-1} = L$; then the conditional mean is

$$E\{\exp[-s\,|\delta_i(\varepsilon)|] \mid L\} = \exp\left\{-L\int_0^\varepsilon [1-\exp(-s\tau)](2\pi\tau^3)^{-1/2}\,d\tau\right\}$$

Recalling that L has an exponential distribution, one gets

$$Ee^{-s_1 L(\varepsilon) - s\delta(\varepsilon)} = \Lambda_\varepsilon\left[\Lambda_\varepsilon + s_1 + \int_0^\varepsilon (1-e^{-s\tau})(2\pi\tau^3)^{-1/2}\,d\tau\right]^{-1} \qquad \text{(A1)}$$

Substitution of Λ_ε and the normalization $L = (\pi\varepsilon/2)^{1/2}\,\eta$, $\delta = \varepsilon\xi^-$, yield (5).

When $s_1 = 0$, (A1) produces the Laplace transform $\varphi(s)$ of ξ^-. Integration by parts in (A1) gives

$$\varphi(s) = \left[1 - \int_0^1 (1-e^{-s\tau})\,d\tau^{-1/2}\right]^{-1} = \left[1 + s\int_0^1 e^{-s\tau}\mu(d\tau)\right]^{-1} \qquad \text{(A2)}$$

where μ is a probability measure on $[0, 1]$ with the density

$$\mu(d\tau)/d\tau = \tau^{-1/2} - 1, \qquad \tau \in (0, 1)$$

Obviously, $1/\varphi(s)$ is an entire function of the complex argument s. Since

$$|1/\varphi - 1| = \left|z\int_0^1 e^{-z\tau}\mu(d\tau)\right| < re^r, \qquad |z| < r$$

it follows that $\varphi(s)$ is regular in the circle $|z| \leqslant r$, provided $r\exp(r) < 1$. Bernstein's inequality yields an exponential estimate for the distribution of ξ^-:

$$P(\xi^- > x) < \varphi(-r)\,e^{-rx}$$

where one may set $r = \kappa > 1/2$ under the condition that $\kappa \exp \kappa < 1$, specifically $\kappa = 0.567$. This proves (7).

Making the change of variables $s\tau = u^2$ in (A2), one gets

$$\varphi(s) = \left[e^{-s} + (2s)^{1/2} \int_0^{(2s)^{1/2}} e^{-u^2/2} \, du \right]^{-1} = (\pi s)^{-1/2}[1 + o(1)], \qquad s \to \infty$$

Therefore, from Tauber's theorem[4] one gets (8). Hence there follows the finiteness of the moments of ξ^- for negative powers $q \in (-1/2, 0)$, i.e.,

$$E[\xi^-]^q = |q| \int_0^\infty x^{q-1} P(\xi^- > x) \, dx < \infty, \qquad q \in (-1/2, 0)$$

Since $P(\xi^- > x)$ exponentially decays, these moments are finite for all $q > 0$.

Relation (3) expresses the obvious circumstance that the set $\bigcup_i (\delta_i(\varepsilon) \cup \Delta_i(\varepsilon))$, $i = 1,..., \nu(t, s)$, must cover $(0, t)$.

The first ε-cluster begins at 0, as can be deduced from Kolmogorov's 0–1 law. Hence the number of ε-clusters that cover $Z \cap [0, t]$ is not less than 1.

Proof of Statement 2. Let

$$\eta_p = \xi^- \eta^p = \eta^p \int_0^1 \tau\pi((0, \eta), dl)$$

One has

$$\begin{aligned} \varphi_p(\theta) &= E \exp(-\theta\eta_p) = E(E \exp(-\theta\eta_p) \mid \eta) \\ &= E \exp\left\{ -\eta \int_0^1 [1 - \exp(-\theta\eta^p x)](2x^{3/2})^{-1} \, dx \right\} \\ &= \int_0^\infty \exp\left\{ -\eta \left(1 - \int_0^1 [1 - \exp(-\theta\eta^p x)] \, dx^{-1/2} \right) \right\} d\eta \end{aligned}$$

Put $\theta = s^{-p}$ and make the change of variables $\eta = su$. One has

$$\varphi_p(\theta) = s \int_0^\infty \exp[-s\psi(u)] \, du \tag{A3}$$

where

$$\begin{aligned} \psi(u) &= u + u^{1+p} \int_0^1 \exp(-u^p \tau) \, \mu(dt) \\ &= u \exp(-u^p) + \sqrt{2}\, u^{p/2} \int_0^{2^{1/2} u^{p/2}} \exp(-x^2/2) \, dx \end{aligned} \tag{A4}$$

with the change $u^p \tau = x^2/2$ in the last equality.

After some elementary algebra we obtain

$$\frac{d}{dn}\psi(u)=\exp(-u^{\rho})+(2u^{\rho})^{1/2}(1+\rho/2)\int_0^{2^{1/2}u^{\rho/2}}\exp(-x^2/2)\,dx$$

If follows that $\psi(u)$ increases from 0 to ∞ for $\rho>-2$.
Using the asymptotic

$$\int_x^{\infty}\exp(-u^2/2)\,du\,e^{(x^2/2)}=x^{-1}-x^{-3}+o(x^{-3}),\qquad x\to\infty$$

One gets

$$\psi(x)=\begin{cases}\pi^{1/2}x^{1+\rho/2}+\frac{1}{2}x^{1-\rho}\exp(-x^{\rho})[1+o(1)], & x^{\rho}\to\infty\\ x+x^{\rho+1}-\frac{1}{6}x^{2\rho+1}+O(x^{3\rho+1}), & x^{\rho}\to 0\end{cases}\tag{A5}$$

We are going to find the asymptotic of $F_{\rho}(x)=P\{\eta_{\rho}<x\}$, $x\to 0$.

Case $\rho>-2$, $x\to 0$. From (A3) one gets

$$\varphi_{\rho}(\theta)=s\int_0^{\infty}e^{-s\psi}\,dx(\psi),\qquad \theta=s^{-\rho}$$

where $x(\psi)$ is an increasing function which is the inverse of $\psi(x)$. From (A5)

$$x(\psi)=(\psi^2/\pi)^{1/(2+\rho)}[1+o(1)],\qquad (\psi\to\infty,\rho>0)\quad\text{or}\quad(\psi\to 0,\rho<0)$$

From Tauber's theorem it follows therefore that

$$\varphi_{\rho}(\theta)=c_{\rho}s^{1-2/(2+\rho)}[1+o(1)]=c_{\rho}\theta^{-1/(2+\rho)}[1+o(1)],\qquad \theta\to\infty\tag{A6}$$

where

$$c_{\rho}=\Gamma(1+2/(2+\rho))\,\pi^{-1/(2+\rho)}$$

Again, the asymptotic (A6) gives

$$F_{\rho}(x)=c_{\rho}/\Gamma(1+1/(2+\rho))]\,x^{1/(2+\rho)}[1+o(1)],\qquad x\downarrow 0\tag{A7}$$

Case $\rho<-2$, $x\to 0$. Let $\rho_1=-2+(2n)^{-1}$, $\rho_2=\rho-\rho_1<0$. The following obvious inequality holds for any pair of nonnegative random variables ξ_1,ξ_2:

$$P(\xi_1\xi_2<x)<P(\xi_1<\sqrt{x})+P(\xi_2<\sqrt{x}),\qquad x>0$$

Set $\xi_1 = \eta^{\rho_1}\xi^-$ and $\xi_2 = \eta^{\rho_2}$, and recall that η has the exponential distribution. Using (A7), we get

$$P(\eta^\rho \xi^- < x) < F_{\rho_1}(x^{1/2}) + \exp(-x^{-1/|2\rho_2|}) = O(x^n), \qquad x \to 0$$

Since $n > 0$ can be chosen arbitrarily, one has

$$F_\rho(x) = o(x^N), \qquad x \to 0, \qquad \forall N > 0, \quad \rho < -2$$

Let us find the asymptotic of $1 - F_\rho(x) = \bar{F}_\rho(x)$ as $x \to 0$.

Case $\rho < -4$, $x \to \infty$. Let

$$I(\theta) = \int_0^\infty e^{-x\theta} \bar{F}_\rho(x)\, dx = \theta^{-1}[1 - \varphi_\rho(\theta)]$$

$$= \theta^{-1} s \int_0^\infty [e^{-sx} - e^{-s\psi(x)}]\, dx = \theta^{-1} s(I_1 + I_2) \tag{A8}$$

where $\theta = s^{-\rho}$, $I_1 + I_2 = (\int_0^\varepsilon + \int_\varepsilon^\infty)[\cdot]\, dx$, and $\varepsilon > 0$ is a small fixed number. The use of (A5) yields

$$I_1 = \int_0^\varepsilon [e^{-sx} - e^{-s\psi(x)}]\, dx$$

$$= \int_0^\varepsilon [1 - e^{-s\psi(x)}]\, dx - \int_0^\varepsilon [1 - e^{-sx}]\, dx$$

$$= \int_0^\varepsilon [1 - \exp(-s' x^{-\rho'})]\, dx + O(s), \qquad s \to 0$$

where $\rho' = -(1 + \rho/2) > 1$ and $s' = \pi^{1/2} s$.

The change of variables $s' x^{-\rho'} = u$ gives

$$I_1 = (s')^{1/\rho'} (\rho')^{-1} \int_{s'/\varepsilon^{\rho'}}^\infty (1 - e^{-u})\, u^{-1-1/\rho'}\, du + O(s)$$

$$= \Gamma(1 - 1/\rho')(\pi^{1/2} s)^{1/\rho'} [1 + o(1)], \qquad s \to 0$$

The second integral is

$$|I_2| = \left| \int_\varepsilon^\infty [e^{-sx} - e^{-s\psi(x)}]\, dx \right| \leqslant s \int_\varepsilon^\infty |(\psi(x) - x)|\, dx$$

Here $\psi(x) - x = \sqrt{\pi}\, x^{1+\rho/2}[1 + o(x^{-1})]$, $x \to \infty$, with $1 + \rho/2 < -1$. Hence $I_2 = O(s) = o(I_1)$, $s \to 0$.

Thus,

$$I(\theta)=\theta^{-1}s(I_1+I_2)=c_\rho s^{-2/(2+\rho)}\theta^{-1}s[1+o(1)]$$
$$=c_\rho\theta^{-(3+\rho)/(2+\rho)}[1+o(1)], \qquad \theta\to 0$$

where $c_\rho=\Gamma((4+\rho)/(2+\rho))\,\pi^{-(2+\rho)^{-1}}$.

By Tauber's theorem

$$\bar{F}_\rho(x)=c_\rho/\Gamma((3+\rho)/(2+\rho))\,x^{1/(2+\rho)}[1+o(1)], \qquad x\to\infty$$

Case $-2>\rho>-4$, $x\to\infty$. Equation (A8) yields

$$I(\theta)=[1-\varphi_\rho(\theta)]/\theta$$
$$=\theta^{-1}s^2\int_0^\infty[\psi(x)-x]\,dx\,[1+o(1)], \qquad s\to 0$$

To see this, it is sufficient to verify that

$$k=\int_0^\infty(\psi(x)-x)\,dx=\int_0^\infty dx\,x^{\rho+1}\int_0^1\exp(-\tau x^\rho)\,d\mu(\tau)$$

is finite. Changing the order of integration, we have

$$k=-\Gamma(1+2/\rho)\,\rho/[(4+\rho)/2]$$

Hence

$$I(\theta)=-\Gamma(2/\rho)/(4+\rho)\,\theta^{-2/\rho-1}[1+o(1)], \qquad \theta\to 0$$

and by Tauber's theorem

$$\bar{F}_\rho(x)=\tfrac{1}{2}\,|\rho|\,(4+\rho)^{-1}\,x^{2/\rho}[1+o(1)], \qquad x\to\infty$$

Case $\rho>-2$, $x\to\infty$. The conditional Laplace transform is

$$E\{\exp(-\theta\eta_\rho)\,|\,\eta\}=\exp\left[\eta\int_0^1\exp(1-\theta\eta^\rho x)\,dx^{-1/2}\right]$$

It follows that the cumulants η_ρ for η fixed are given by

$$\kappa_r=-\eta\int_0^1(\eta^\rho x)^r\,dx^{-1/2}=\eta^{r\rho+1}/(2r-1), \qquad r=1,2,\ldots$$

In that case the conditional moments of η_ρ are

$$E\{\eta_\rho^n \mid \eta\} = \sum_{i_1 + \cdots + i_n = n} c_{i_1 \ldots i_n} \kappa_{i_1} \cdots \kappa_{i_n} = \sum_{i=1}^{n} \eta^{n\rho + i} c_i, \qquad c_i > 0$$

Hence the unconditional moments $E\eta_\rho^n = m_n(\rho)$ are finite if $E\eta^{n\rho+1} < \infty$ and $E\eta^{n(\rho+1)} < \infty$. The variable η is exponentially distributed, so that

$$m_n(\rho) < \infty \Leftrightarrow n\rho + 2 > 0$$

When $\rho > 0$, all moments exist; otherwise

$$m_n(\rho) < \infty, \qquad m_{n+1}(\rho) = \infty \Leftrightarrow \rho \in (-2/n, -2/(n+1)] \tag{A9}$$

that is, when $p \in I_n = (-2/n, -2/(n+1))$, exactly n moments of η_ρ are finite.

According to (A9), the function $\varphi_\rho(\theta)$ can be differentiated n times, when $\rho \in I_n$.

In virtue of

$$s^{2 - 2/(n+1)} < s^{-n\rho} = \theta^n < s^2, \qquad \rho \in I_n, \qquad |s| < 1$$

$\varphi_\rho(\theta)$ can be expanded around zero

$$\varphi_\rho(\theta) = \sum_{k=0}^{n} \frac{(-\theta)^k}{k!} m_k(\rho) + c_n s^2[1 + o(1)], \qquad \theta = s^{-\rho} \tag{A10}$$

To see this, let $\rho \in I_n$. By (A3) and (A4)

$$\varphi_\rho(\theta) = \int_0^\infty s \exp(-sx) \exp[-sx^{1+\rho} \psi_1(x^\rho)]\, dx = 1 + \sum_{k=1}^{n} A_k + R_n \tag{A11}$$

where

$$\psi_1(u) = \int_0^1 e^{-u\tau}\, d\mu(\tau) \leqslant 1$$

and

$$A_k = \int_0^\infty s e^{-sx} (-sx^{1+\rho})^k\, \psi_1^k(x^\rho)\, dx/k!, \qquad k = 1, \ldots, n$$

The remainder R_n can obviously be evaluated by

$$\begin{aligned} R_n &< \int_0^\infty se^{-sx}[sx^{1+\rho}\psi_1(x^\rho)]^{n+1}\,dx/(n+1)! \\ &< \int_0^\infty se^{-sx}(sx^{1+\rho})^{n+1}\,dx/(n+1)! \\ &= cs^{\rho(n+1)} = o(s^2) \end{aligned} \tag{A12}$$

where

$$c = \Gamma((1+\rho)(n+1)+1)/\Gamma(n+2), \qquad n \geqslant 1$$

To evaluate A_k note that

$$\begin{aligned} \psi_1^k(u) &= \int_0^k e^{-u\tau}\,d\mu^{(k)}(\tau) \\ &= \sum_{p=0}^{N} \frac{(-u)^p}{p!}\mu_p^{(k)}(\rho) + \theta_N \frac{k^{N+1}}{(N+1)!}u^{N+1} \end{aligned} \tag{A13}$$

where $|\theta_N| < 1$ and $\mu^{(k)}$ is the kth convolution of measure μ; $\mu_p^{(k)}$ are moments of $\mu^{(k)}$.

One has

$$\begin{aligned} k!\,A_k = {} & \int_0^\infty se^{-sx}(-sx^{1+\rho})^k \left[\psi_1(x^\rho) - \sum_{p=0}^{n-k} \mu_p^{(k)}(-x^\rho)^p/p!\right] dx \\ & + \sum_{p=0}^{n-k} \int_0^\infty se^{-sx}(-sx^{1+\rho})^k\,(-x^\rho)^p\,\mu_p^{(k)}\,dx/p! \end{aligned}$$

The first term A_{k1} in A_k is of order s^2 when $k=1$ and $o(s^2)$ when $k>1$. To see this, evaluate A_k, $k>1$, using (A13):

$$\begin{aligned} A_{k1} &\leqslant c\int_0^\infty se^{-sx}(-sx^{1+\rho})^k\,x^{\rho(n-k+1)}\,dx \\ &= c\Gamma(\rho(n+1)+k+1)\,s^{\rho(n+1)} = o(s^2) \end{aligned} \tag{A14}$$

When $s=0$, the integrand is

$$\begin{aligned} &O(x^{1+\rho(n+1)}) \qquad \text{as} \quad x \to \infty \\ &O(x^{1+\rho n}) \qquad \text{as} \quad x \to 0 \end{aligned}$$

Both singularities are integrable for $\rho \in I_n$, so

$$A_{k1} = -s^2 \int_0^\infty x^{1+\rho} \left(\psi_1(x^\rho) - \sum_{p=0}^{n-1} \mu_p^{(k)} (-x^\rho)^p/p! \right) dx\, [1 + o(1)] \quad \text{(A15)}$$

The other components A_k can be explicitly expressed in terms of the gamma function. Hence

$$A_k = A_{k1} + \sum_{p=0}^{n-k} \frac{(-\theta)^{k+p}}{k!\, p!} \mu_p^{(k)} \Gamma(k + 1 + \rho(k+p)) \quad \text{(A16)}$$

Substitution of (A12) and (A14)–(A16) in (A11) yields (A10).

Let

$$\bar{F}_n(x) = \int_x^\infty (t - x)^n/n!\, dF_\rho(t)$$

Since

$$\varphi_\rho(x) = \sum_{k=0}^{n} \frac{(-\theta)^k}{k!} m_k(\rho) + \frac{(-\theta)^{n+1}}{(n+1)!} \int_0^\infty e^{-\theta x} \bar{F}_n(x)\, dx$$

(A10) shows that

$$\int_0^\infty e^{-\theta x} \bar{F}_n(x)\, dx = c_n \theta^{-2/\rho - n - 1} [1 + o(1)], \qquad \theta \to 0$$

where $n < -2/\rho < n + 1$. By Tauber's theorem

$$\bar{F}_n(x) = \tilde{c}_n x^{2/\rho + n} [1 + o(1)], \qquad x \to \infty$$

The use of L'Hôspital's rule gives

$$\begin{aligned}
\tilde{c}_n &= \lim_{x \to \infty} \bar{F}_n(x)/x^{2/\rho + n} \\
&= \lim_{x \to \infty} D^{(n)} \bar{F}_n(x)/D^{(n)} x^{2/\rho + n} \\
&= (-1)^n \, [(2/\rho + n) \cdots (2/\rho + 1)]^{-1} \lim_{x \to \infty} \bar{F}_\rho(x)/x^{2/\rho}
\end{aligned}$$

where $D^{(n)} = (d/dx)^n$.

Hence, $1 - F_\rho(x) = \hat{c}_n x^{2/\rho} [1 + o(1)]$.

Proof of Statement 3. Let us fix t and put $\nu_\varepsilon = \nu(t/\varepsilon)$ and $\nu_\varepsilon^* = \nu^*(t/\varepsilon)$. We have

$$P_n = P\{\nu_\varepsilon^* - \nu_\varepsilon \geqslant n\} = P\left\{\sum_{i=1}^{n} \xi_{\nu_\varepsilon+i}^{+} < \sum_{i=1}^{\nu_\varepsilon} \xi_i^{-}\right\}$$

The moment ν_ε is Markovian, that is, the event $\{\nu_\varepsilon = m\}$ is measurable with respect to $\{\xi_i^{\pm}, i = 1,..., m\}$. Hence

$$P_n = P\left\{\sum_{i=1}^{n} \xi_i^{+} < \sum_{i=1}^{\nu_\varepsilon} \xi_i^{-}\right\}$$

where $\{\xi_i^{+}\}$ are independent of ν_ε and $\{\xi_i^{-}\}$. Let

$$B_\rho = \left\{\omega \colon \sum_{i=1}^{\nu_\varepsilon} \xi_i^{-} < \varepsilon^{-1/2-\rho}\right\}$$

Then

$$P(\bar{B}_\rho) = o(1), \qquad \rho > 0; \qquad P(\bar{B}_\rho) = o(1), \qquad \rho < 0 \tag{A17}$$

In fact,

$$\begin{aligned} P(\bar{B}_\rho) &< P\left\{\sum_{i=1}^{\nu_\varepsilon} \xi_i^{-} > \varepsilon^{-1/2-\rho}, \nu_\varepsilon > \varepsilon^{-\theta}\right\} + P\{\nu_\varepsilon < \varepsilon^{-\theta}\} \\ &< P\left\{\frac{1}{n_\varepsilon} \sum_{1 \leqslant i \leqslant n_\varepsilon} \xi_i^{-} > \varepsilon^{-1/2-\rho+\theta}\right\} + P\{\nu_\varepsilon < n_\varepsilon\}, \qquad n_\varepsilon = \varepsilon^{-\theta} \end{aligned}$$

By the law of large numbers

$$n^{-1} \sum_{i=1}^{n} \xi_i^{-} \xrightarrow{d} E\xi^{-} = 1, \qquad n \to \infty$$

and according to P. Levy (see ref. 10)

$$d\text{-}\lim_{\varepsilon \to 0} (\pi\varepsilon/2)^{1/2} \nu(t/\varepsilon) = L(t) \tag{A18}$$

Hence, choosing $1/2 < \theta < 1/2 + \rho$, we have $P(\bar{B}_\rho) = o(1)$, $\rho > 0$. Similarly it is proved that $P(B_\rho) = o(1)$, $\rho < 0$. Find an upper bound of p_n. Let $p > 0$; then

$$\begin{aligned} p_{n_\varepsilon} &\leqslant P\left(\sum_{1 \leqslant i \leqslant n_\varepsilon} \xi_i^{+} < \sum_{1 \leqslant i \leqslant \nu_\varepsilon} \xi_i^{-}, B_\rho\right) + P(\bar{B}_\rho) \\ &\leqslant P\left(n_\varepsilon^{-2} \sum_{1 \leqslant i \leqslant n_\varepsilon} \xi_i^{+} < \varepsilon^{-1/2-\rho+2\theta}\right) + P(\bar{B}_\rho), \qquad n_\varepsilon = \varepsilon^{-\theta} \end{aligned}$$

One has $\xi_i^+ \in G_{1/2}$; hence

$$\bar{S}_n(\xi^+) := n^{-2} \sum_{i=1}^{n} \xi_i^+ \to g_{1/2}(c)$$

Recalling (A17), one has $p_{n_\varepsilon} = o(1)$, $\varepsilon \to 0$, if $2\theta - 1/2 - \rho > 0$ and $\rho > 0$. Hence

$$P(v_\varepsilon^* - v_\varepsilon > \varepsilon^{-\theta}) = o(1), \qquad \theta > 1/4$$

Let us find a lower bound of p_n. Let $\rho < 0$; then

$$\begin{aligned} p_{n_\varepsilon} &\geqslant P\left(\sum_{1 \leqslant i \leqslant n_\varepsilon} \xi_i^+ < \sum_{1 \leqslant i \leqslant v_\varepsilon} \xi_i^-, \bar{B}_\rho \right) \\ &\geqslant P(\bar{S}_n(\xi^+) < \varepsilon^{-1/2 - \rho + 2\theta}, \bar{B}_\rho) \\ &= P\{\bar{S}_n(\xi^+) < \varepsilon^{-1/2 - \rho + 2\theta}\}\ P(\bar{B}_\rho) \end{aligned}$$

Relations (A17) and $\xi^+ \in G_{1/2}$ thus yield

$$P(v_\varepsilon^* - v_\varepsilon < \varepsilon^{-\theta}) = 1 - P_{n_\varepsilon} = o(1), \qquad \varepsilon \to 0, \qquad \theta < 1/4$$

The limit distribution of $v^*(t/\varepsilon)\, \varepsilon^{1/2}$ follows from (A18) and the closeness of $v^*(t/\varepsilon)$ and $v(t/\varepsilon)$.

Proof of Statement 4. We begin by proving that (10) holds when $v(t/\varepsilon)$ is replaced by $v^*(t/\varepsilon)$. Fix the times $0 < t_1 \cdots < t_N$. Let

$$\begin{aligned} \xi_\varepsilon^*(t) &= \varepsilon^{1/2\alpha} \sum_{i=1}^{v^*(t/\varepsilon)} \zeta_i [\sqrt{\varepsilon}\, v^*(t/\varepsilon)]^\beta \\ \Delta v_k^* &= v^*(t_k/\varepsilon) - v^*(t_{k-1}/\varepsilon), \qquad t_{-1} = 0 \qquad\qquad \text{(A19)} \\ \varphi(\theta) &= E \exp(-\theta\zeta) \end{aligned}$$

Using the independence of $v^*(t)$ and $\{\zeta_i\}$, we can find the Laplace transform for the variables $\{\xi_i^*(t_i), i = 1,..., N\}$:

$$\Phi_\varepsilon := E \exp\left(-\sum_{i=1}^{N} \theta_i \xi_i^*(t_i) \right) = E \exp\left(-\sum_{k=1}^{N} \psi_{\varepsilon,k} \sqrt{\varepsilon}\, \Delta v_k^* \right)$$

where

$$\psi_{\varepsilon,k} = -\ln \left\{ \varphi\left(\sum_{i=k}^{N} \theta_i [\sqrt{\varepsilon}\, v^*(t_i/\varepsilon)]^\beta n_\varepsilon^{-1/\alpha} \right) \right\}^{n_\varepsilon}, \qquad n_\varepsilon = \varepsilon^{-1/2}$$

In virtue of (9) the ζ_i belong to the attraction region G_a. Hence

$$n^{-1/\alpha} \sum_{i=1}^{n} \zeta_i \xrightarrow{d} g_\alpha(c), \qquad n \to \infty \tag{A20}$$

Accordingly, the Laplace transforms of these distributions converge uniformly on any finite interval $0 < \lambda < \Lambda$:

$$[\varphi(\lambda n^{-1/\alpha})]^n \to \exp(-\lambda^\alpha c), \qquad n \to \infty$$

By Statement 3,

$$\sum_{i=k}^{N} \theta_i [\sqrt{\varepsilon}\, v^*(t_i/\varepsilon)]^\beta \xrightarrow{d} \sum_{i=k}^{N} \theta_i [(2/\pi)^{1/2} L(t_i)]^\beta$$
$$\sqrt{\varepsilon}\, \Delta v_k^* \xrightarrow{d} (2/\pi)^{1/2} [L(t_k) - L(t_{k-1})]$$

hence

$$\psi_{\varepsilon,k} \xrightarrow{d} c \left\{ \sum_{i=k}^{N} \theta_i [(2/\pi)^{1/2} L(t_i)]^\beta \right\}^\alpha$$

and

$$\psi_\varepsilon := \sum_{k=1}^{N} \psi_{\varepsilon,k}(\sqrt{\varepsilon}\, \Delta v_k^*)$$
$$\xrightarrow{d} c \sum_{k=1}^{N} \left\{ \sum_{i=k}^{N} \theta_i [(2/\pi)^{1/2} L(t_i)]^\beta \right\}^\alpha (2/\pi)^{1/2} [L(t_k) - L(t_{k-1})] = \psi_0$$

Recalling that $\exp(-\psi_\varepsilon) \leqslant 1$, one has

$$\Phi_\varepsilon = E \exp(-\psi_\varepsilon) \to E \exp^{(-\psi_0)} = \Phi_0$$

The resulting limit is the Laplace transform for the distribution of the random vector

$$[(2/\pi)^{1/2} L(t_i)]^\beta\, g_\alpha(c(2/\pi)^{1/2} L(t_i)), \qquad k = 1,..., N$$

where $g_\alpha(u)$ is a Levy process with an exponent α that is independent of $L(t)$.

We now return to the original prelimiting process (10). It is obtained from (A19) by omitting the asterisk. One gets

$$\xi_\varepsilon(t) = \xi_\varepsilon^*(t)(v_\varepsilon / v_\varepsilon^*)^\beta - \Omega_\varepsilon$$

where

$$\Omega_\varepsilon = \varepsilon^{1/(2\alpha)} \sum_{1 \leqslant i \leqslant \nu_\varepsilon^* - \nu_\varepsilon} \zeta_{\nu_\varepsilon + i} (\sqrt{\varepsilon}\, \nu_\varepsilon)^\beta$$

and

$$\nu_\varepsilon^* = \nu^*(t/\varepsilon), \qquad \nu_\varepsilon = \nu(t/\varepsilon)$$

Obviously

$$\nu_\varepsilon^*/\nu_\varepsilon = (\nu_\varepsilon^* - \nu_\varepsilon)\sqrt{\varepsilon}/(\nu_\varepsilon \sqrt{\varepsilon}) + 1 \xrightarrow{d} 1$$

since

$$(\nu_\varepsilon^* - \nu_\varepsilon)\sqrt{\varepsilon} \xrightarrow{d} 0 \qquad \text{and} \qquad \nu_\varepsilon \sqrt{\varepsilon} \xrightarrow{d} (2/\pi)^{1/2} L(t)$$

For the same reason

$$\Omega_\varepsilon = \Omega'_\varepsilon [(\nu_\varepsilon^* - \nu_\varepsilon)\sqrt{\varepsilon}]^{1/\alpha} [\sqrt{\varepsilon}\, \nu_\varepsilon]^\beta \xrightarrow{d} 0$$

Hence

$$\Omega'_\varepsilon = [(\nu_\varepsilon^* - \nu_\varepsilon)]^{-1/\alpha} \sum_{1 \leqslant i \leqslant \nu_\varepsilon^* - \nu_\varepsilon} \zeta_{\nu_\varepsilon + i}$$

Since the moment ν_ε is Markovian,

$$\Omega'_\varepsilon \stackrel{d}{=} \kappa^{-1/\alpha} \sum_{i=1}^{\kappa} \zeta_i, \qquad \kappa = \nu_\varepsilon^* - \nu_\varepsilon$$

where κ and $\{\xi_i\}$ are independent.

In virtue of the above results

$$\Omega'_\varepsilon \xrightarrow{d} g_\alpha(c), \qquad \varepsilon \to 0$$

because $\nu_\varepsilon^* - \nu_\varepsilon \to \infty$, $\varepsilon \to 0$. Hence

$$d\text{-lim}\, \xi_\varepsilon(t) = d\text{-lim}\, \xi'_\varepsilon(t), \qquad \varepsilon \to 0$$

The case $E\zeta = m < \infty$ can be treated similarly, the only difference being that (A20) is replaced by the law of large numbers:

$$n^{-1} \sum_{i=1}^{n} \zeta_i \xrightarrow{d} m$$

and $\alpha = 1$, $c = m$, $g_\alpha(u) \equiv u$.

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